

Symmetric Calorons

R. S. Ward*

Department of Mathematical Sciences,
University of Durham,
Durham DH1 3LE

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Abstract

Calorons (periodic instantons) interpolate between monopoles and instantons, and their holonomy gives approximate Skyrmon configurations. We show that, for each caloron charge $N \leq 4$, there exists a one-parameter family of calorons which are symmetric under subgroups of the three-dimensional rotation group. In each family, the corresponding symmetric monopoles and symmetric instantons occur as limiting cases. Symmetric calorons therefore provide a connection between symmetric monopoles, symmetric instantons and Skyrmons.

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*email: richard.ward@durham.ac.uk

1 Introduction

Calorons are finite-action self-dual gauge fields in four dimensions, which are periodic in one of the four coordinates. Call the periodic coordinate t , with period β . Special cases include instantons on \mathbf{R}^4 (where $\beta \rightarrow \infty$) and BPS monopoles (where the gauge field is independent of t). The holonomy Ω of the gauge field in the t -direction is a map from \mathbf{R}^3 to the gauge group, and as such can serve as an approximation to Skyrmons [1]. Calorons therefore provide a link between monopoles, instantons and Skyrmons.

Skyrmions resemble polyhedral shells, invariant under appropriate subgroups of the three-dimensional rotation group $O(3)$ [2, 3]. The idea of producing approximate Skyrmon configurations as instanton holonomy has motivated several studies of instantons invariant under such groups [4, 5, 6, 7]. Finally, there are symmetric monopoles [8] which have the same polyhedral shape as the Skyrmons of corresponding charge, suggesting a kinship between Skyrmons and monopoles [9]. So symmetric calorons, namely calorons invariant under subgroups G of $O(3)$ (rotations about the t -axis), are relevant in this context. This Letter demonstrates the existence of symmetric calorons of charge N , for $N \leq 4$; they include, as limiting cases, symmetric monopoles and symmetric instantons.

Large classes of calorons were described some years ago [10, 11, 12, 13]; of these, only the $N = 1$ case admits the relevant symmetry. So one needs more general solutions. There is a construction (the ADHMN construction) which generates caloron solutions [14], possibly all of them (see [15] for a recent analysis). In the last few years, this construction has been used to investigate and interpret caloron solutions, especially those for which the holonomy Ω is non-trivial at spatial infinity [16, 17, 18]; but this recent work was not concerned with symmetric solutions as such. In this Letter, we shall see how symmetric calorons arise from the ADHMN construction.

2 Calorons, Monopoles and Skyrmons

We take the gauge group to be $SU(2)$ throughout. The standard coordinates on \mathbf{R}^4 are denoted $x^\mu = (x^1, x^2, x^3, x^4) = (x^j, t)$; let r be the quantity defined by $r^2 = x^j x^j$. The gauge potential A_μ is anti-hermitian, and the corresponding gauge field is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. A gauge transformation acts as $A_\mu \mapsto \Lambda^{-1} A_\mu \Lambda + \Lambda^{-1} \partial_\mu \Lambda$. A *caloron* [10, 11, 12] is a gauge field with the following properties:

- $A_\mu(x^\alpha)$ is periodic in $x^4 = t$, with period β (in some gauge);
- $A_\mu(x^\alpha)$ is smooth everywhere (in some gauge);

- $F_{\mu\nu}$ is self-dual: $F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$;
- $\text{tr}(F_{\mu\nu}F_{\mu\nu}) = O(1/r^4)$ as $r \rightarrow \infty$.

A special case of this is where A_μ is independent of $x^4 = t$; this is a *monopole*, where we make the usual interpretation of A_t as a Higgs field Φ . The holonomy (or Wilson loop)

$$\Omega(x^j) = \mathcal{P} \exp \left[- \int_0^\beta A_t(x^j, t) dt \right] \quad (1)$$

in the t -direction takes values in the gauge group; under a periodic gauge transformation, it transforms as

$$\Omega(x^j) \mapsto \Lambda(x^j, 0)^{-1} \Omega(x^j) \Lambda(x^j, 0). \quad (2)$$

The quantity $\Omega(x^j)$ is, in general, non-trivial at spatial infinity [11]; but for the examples below, $\Omega(x^j)$ tends to a constant group element (in fact the identity) as $r \rightarrow \infty$. Such a field may be viewed as an approximate *Skyrmion* configuration; the Skyrme number is the degree of Ω , and the normalized Skyrme energy is

$$E = \frac{1}{12\pi^2} \int \left\{ -\frac{1}{2} \text{tr}(L_j L_j) - \frac{1}{16} \text{tr}([L_i, L_j][L_i, L_j]) \right\} d^3x, \quad (3)$$

where $L_j = \Omega^{-1} \partial_j \Omega$. Provided Ω is asymptotically trivial, the topological charge (caloron number)

$$N = -\frac{1}{32\pi^2} \int_0^\beta dt \int d^3x \text{tr}(\varepsilon_{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}) \quad (4)$$

is an integer, and is equal to the Skyrme number of Ω [11]. In the t -independent (monopole) case, it is also the monopole number, provided we take β to be related to the asymptotic norm of the Higgs field by

$$-\frac{1}{2} \text{tr}(\Phi_\infty)^2 = \left(\frac{\pi}{\beta} \right)^2. \quad (5)$$

A large number of caloron solutions can be generated [10] by the Corrigan-Fairlie-'tHooft [19] or Jackiw-Nohl-Rebbi [20] ansatz. These express the gauge potential in terms of a solution ϕ (periodic, in the caloron case) of the four-dimensional Laplace equation. For example, the component A_t is given by

$$A_t = \frac{i}{2} (\partial_j \log \phi) \sigma_j \quad (6)$$

where σ_j are the Pauli matrices. For the JNR solutions one has $\phi \rightarrow 0$ as $r \rightarrow \infty$, whereas for the CF'tH solutions one has $\phi \rightarrow 1$ as $r \rightarrow \infty$. In the case of instantons on \mathbf{R}^4 , one regards the CF'tH solutions as being limiting cases of the JNR solutions,

but for calorons it is the other way round: to produce an N -caloron in JNR form, one uses a ϕ with N poles (not $N + 1$ as for instantons), and this is a limiting case of the CF'tH form with N poles.

To illustrate this, let us review the $N = 1$ case. The 1-caloron (with trivial holonomy at infinity) is generated [10] by the 1-pole function

$$\phi = 1 + \frac{W^2 \sinh(\mu r)}{2r[\cosh(\mu r) - \cos(\mu t)]}, \quad (7)$$

where $\mu = 2\pi/\beta$, and $W > 0$ is a constant. This caloron is spherically-symmetric; it depends on the period β and on the parameter W . The gauge field is not affected by an overall scale factor in ϕ , so the $W \rightarrow \infty$ limit of (7) gives, in effect, the JNR-type solution with

$$\phi = \frac{\sinh(\mu r)}{2r[\cosh(\mu r) - \cos(\mu t)]}; \quad (8)$$

this corresponds to a 1-caloron which is in fact gauge-equivalent to the 1-monopole [21]. Another way of viewing things is to use the dimensionless combination $\theta = \beta/W^2$: for $\theta = 0$ (or $W \rightarrow \infty$) we get the 1-monopole, while for $\theta \rightarrow \infty$ (or $\beta \rightarrow \infty$) we get the 1-instanton on \mathbf{R}^4 . In other words, we have a one-parameter family of spherically-symmetric calorons, with the 1-monopole at one end and the 1-instanton at the other end. The holonomy $\Omega(x^j)$ can be computed exactly in this case [22, 23]; if one restricts to spherically-symmetric gauges, then Ω is actually gauge-invariant. The Skyrme energy (3) of this configuration Ω attains a minimum for $\theta \approx 7$; this minimum is only slightly less [22] than the value obtained from 1-instanton holonomy.

It is straightforward to produce spherically-symmetric calorons of higher charge in this way: for example, the function

$$\phi = 1 + \frac{W^2 \sinh(\mu r)}{2r[\cosh(\mu r) - \cos(\mu t)]} + \frac{\widehat{W}^2 \sinh(\mu r)}{2r[\cosh(\mu r) - \cos(\mu(t - t_0))]} \quad (9)$$

generates a spherically-symmetric 2-caloron, for any $t_0 \in (0, \beta)$ and $W, \widehat{W} > 0$. The holonomy of this is a spherically-symmetric (hedgehog) 2-Skymion configuration (cf. [1, 4]). The limits $\beta \rightarrow \infty$ and $W, \widehat{W} \rightarrow \infty$ are both regular; the former is a 2-instanton, but the latter is not a 2-monopole (since, unlike in the $N = 1$ case, the t -dependence cannot be gauged away). It seems very unlikely that the CF'tH ansatz can yield any examples (other than for $N = 1$) of symmetric calorons having symmetric monopoles as a limiting case — for that, one needs more general solutions. A way of generating such solutions is described in the next section.

3 The ADHMN Construction for Calorons

There is a construction which produces caloron solutions [14]; for gauge group $SU(2)$, and for calorons which have trivial holonomy at infinity, it is as follows. As before, N is a positive integer which will turn out to be the caloron charge, and β is a positive number which will turn out to be the caloron period. It is convenient to use quaternion notation, with a quaternion q being represented by the 2×2 matrix $q^4 + i q^j \sigma^j$; in particular, x^μ corresponds to the quaternion $x = t + i x^j \sigma^j$. The unit quaternion ($q^4 = 1, q^j = 0$) is denoted $\mathbf{1}$.

The Nahm data consists of four hermitian $N \times N$ matrix functions $T_\mu(s)$, and an N -row-vector W of quaternions, such that $T_\mu(s)$ is periodic in the real variable s with period $2\pi/\beta$, and the Nahm equation

$$\frac{d}{ds} T_j - i[T_4, T_j] - \frac{i}{2} \varepsilon_{jkl} [T_k, T_l] = \frac{1}{2} \text{tr}_2 (\sigma_j W^\dagger W) \delta(s - \pi/\beta) \quad (10)$$

is satisfied. The trace is over quaternions, so the right-hand-side is an $N \times N$ hermitian matrix (as is the left-hand-side). Given such data, we construct a caloron as follows. Let $U(s, x)$ be an N -column-vector of quaternions, and $V(x)$ a single quaternion, such that

1. $U(s, x)$ is periodic in s with period $2\pi/\beta$;
2. $U(s, x + \beta) = U(s, x) \exp(i\beta s)$;
3. $V(x + \beta) = V(x)$;
4. $\int_{-\pi/\beta}^{\pi/\beta} U(s, x)^\dagger U(s, x) ds + V(x)^\dagger V(x) = \mathbf{1}$;
5. U and V satisfy the linear equation

$$\frac{d}{ds} U - \left[i(T_4 + t I_n) \otimes \mathbf{1} + I_n \otimes x^j \sigma^j + T_j \otimes \sigma^j \right] U = i W^\dagger V \delta(s - \pi/\beta). \quad (11)$$

Note that both T_j and U are periodic in s , and have jump discontinuities at one value of s , which we have taken to be $s = \pi/\beta$. The discontinuities could equally well be located anywhere else; the choice in (10) and (11) is for later convenience. Note also that the overall quaternionic phase of the N -vector $W = [W_1 \dots W_N]$ is irrelevant; so we may, without loss of generality, take W_1 to be real.

The pair (U, V) determines the caloron gauge potential according to

$$A_\mu = V(x)^\dagger \partial_\mu V(x) + \int_{-\pi/\beta}^{\pi/\beta} U(s, x)^\dagger \partial_\mu U(s, x) ds. \quad (12)$$

The freedom in (U, V) is $U \mapsto U\Lambda$, $V \mapsto V\Lambda$, where Λ is a quaternion satisfying $\Lambda^\dagger \Lambda = 1$; this corresponds exactly to the gauge freedom in A_μ .

By contrast, the usual formulation of the ADHMN construction for monopoles involves three matrices $T_j(s)$, satisfying

$$\frac{d}{ds}T_j - \frac{i}{2}\varepsilon_{jkl}[T_k, T_l] = 0. \quad (13)$$

In this case, the $T_j(s)$ are not periodic in s , but rather are smooth on the open interval $|s| < 1$, with poles at the endpoints $s = \pm 1$. (The length of this interval sets the scale of the monopole.) In addition, the T_j satisfy

$$T_j(-s) = T_j(s)^t. \quad (14)$$

The idea here is that given a solution of the monopole Nahm equation (13), one may re-interpret it as a solution of the caloron Nahm equation (10), with $T_4 = 0$ and with a suitable choice of W , namely such that

$$T_j(-\pi/\beta) - T_j(\pi/\beta) = \frac{1}{2} \text{tr}_2 \left(\sigma_j W^\dagger W \right). \quad (15)$$

We need to take $\beta > \pi$, so that the T_j are bounded for $|s| \leq \pi/\beta$. The symmetric part of T_j can, because of (14), be regarded as a continuous periodic function on $[-\pi/\beta, \pi/\beta]$; while the antisymmetric part of T_j has a jump discontinuity as in (15).

The limit $\beta \rightarrow \pi$ is the original monopole, while the limit $\beta \rightarrow \infty$ gives an instanton on \mathbf{R}^4 . This instanton limit works as follows. For $\beta \gg \pi$, we are solving (11) on the small interval $|s| \leq \pi/\beta$, so we may approximate the solution as $U(s) = U_0 + U_1 s$. Equation (15) then gives

$$U_1 = (it + x^j \sigma^j + \mathcal{T}_j \otimes \sigma^j) U_0 = -\frac{i\beta}{2\pi} W^\dagger V, \quad (16)$$

where $\mathcal{T}_j = T_j(0)$, and where U_0 and V satisfy the constraint

$$U_0^\dagger U_0 + V(x)^\dagger V(x) = \mathbf{1}. \quad (17)$$

If we write $\Lambda = \sqrt{\beta/2\pi} W$, then this is exactly the ADHM construction [24] for instantons, with the ADHM matrix Δ being given by

$$\Delta = \begin{bmatrix} \Lambda \\ x + i\mathcal{T}_j \otimes \sigma^j \end{bmatrix}. \quad (18)$$

This Δ is an $(n+1) \times n$ matrix of quaternions, satisfying the condition that $\Delta^\dagger \Delta$ is an $n \times n$ real matrix.

Let us now consider calorons which are symmetric under subgroups of the three-dimensional rotation group acting on x^j . For any rotation R , let $R_2 \in \text{SU}(2)$ denote the image of R in the 2-dimensional irreducible representation of $\text{SO}(3)$; in other

words, R acts on the quaternion x according to $x \mapsto R_2^{-1}xR_2$. Similarly, let R_N denote the image of R in the N -dimensional irreducible representation of $\text{SO}(3)$, and write $\Theta_R = R_N \otimes R_2$. A monopole is invariant [8] under the group $G \subseteq \text{SO}(3)$ iff

$$\Theta_R^{-1}(T_j \otimes \sigma^j) \Theta_R = T_j \otimes \sigma^j \quad (19)$$

for all $R \in G$. For the corresponding caloron to be G -invariant, we need an additional condition on W , and this is easily seen (from (10) and (11)) to be

$$\Theta_R W^\dagger = W^\dagger \tau_R \quad (20)$$

where τ_R , for each $R \in G$, is some quaternionic phase (namely a quaternion with $\tau_R^\dagger \tau_R = \mathbf{1}$). So given a symmetric monopole, there is a family of symmetric calorons parametrized by the solutions W (if there are any) of (15) and (20). In the $N = 1$ case, for example, we have $G = \text{SO}(3)$ (spherical symmetry) and $T_j = 0$; and W is an arbitrary positive constant, which is precisely the parameter appearing in the expression (7). In the next section, we shall see that analogous one-parameter families of symmetric calorons exist for $N = 2, 3$ and 4.

4 Symmetric Examples for $N = 2, 3, 4$

We begin with the $N = 2$ case, taking $G = \text{SO}(2)$ (corresponding to rotations about the x^2 -axis). The solution of (13) which generates the axially-symmetric $N = 2$ monopole is $T_j(s) = f_j(s)\sigma_j$ (not summed over j), where

$$f_1 = f_3 = \frac{\pi}{4} \sec(\pi s/2), \quad f_2 = -\frac{\pi}{4} \tan(\pi s/2). \quad (21)$$

Then (15) and (20) have a solution W which is unique (given that W_1 is real), namely

$$W = \lambda [\mathbf{1} \quad -i\sigma_2], \quad \text{where} \quad \lambda = \sqrt{\frac{\pi}{2} \tan\left(\frac{\pi^2}{2\beta}\right)}. \quad (22)$$

So we get a family of $N = 2$ axially-symmetric caloron solutions, depending on the parameter $\beta > \pi$. It is possible to solve (11) analytically, and hence obtain exact expressions for the caloron (cf. [25] for the monopole case), although the expressions are rather complicated. The limit $\beta \rightarrow \pi$ is the 2-monopole, and $\beta \rightarrow \infty$ is a 2-instanton on \mathbf{R}^4 , generated by the ADHM matrix

$$\Delta = \frac{\pi}{4} \begin{bmatrix} \sqrt{2} & -i\sqrt{2}\sigma_2 \\ i\sigma_3 & i\sigma_1 \\ i\sigma_1 & -i\sigma_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ x & 0 \\ 0 & x \end{bmatrix}. \quad (23)$$

This axially-symmetric 2-instanton can be obtained in the JNR form, and its holonomy was used to approximate the minimum-energy 2-Skyrmion [26, 4]. The holonomy Ω of the caloron gives a one-parameter family of axially-symmetric 2-Skyrmion configurations; as in the $N = 1$ case, this gives an approximation to the true Skyrmion which is better than the instanton one, but only marginally so.

Let us now consider the $N = 3$ case. There is a 3-monopole with tetrahedral symmetry [8, 27], corresponding to the following Nahm data. (Note that the T_j in [8, 27] have to be multiplied by a factor of $-i$ to agree with the conventions used here.) Define

$$\Sigma_1 = 2i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Sigma_2 = 2i \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \Sigma_3 = 2i \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (24)$$

and

$$S_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (25)$$

Then $T_j(s) = x(s)\Sigma_j + y(s)S_j$, where

$$x(s) = -\frac{\omega \wp'(u)}{12 \wp(u)}, \quad y(s) = -\frac{\omega}{\sqrt{3} \wp(u)}, \quad (26)$$

with $u = \omega(s + 3)/3$ and $\omega = \Gamma(1/6)\Gamma(1/3)/(4\sqrt{\pi})$. Here \wp is the Weierstrass p-function satisfying $\wp'(u)^2 = 4\wp(u)^3 + 4$. The unique solution of (15), with $W_1 > 0$, is

$$W = \lambda[\mathbf{1} \quad i\sigma_3 \quad -i\sigma_2], \quad \text{where } \lambda = 2\sqrt{x(\pi/\beta)}. \quad (27)$$

Explicit calculation then verifies that (20) is satisfied for each of the elements of the tetrahedral group. So we have a one-parameter family of tetrahedrally-symmetric 3-calorons, interpolating between the tetrahedral 3-monopole and a tetrahedrally-symmetric 3-instanton. The latter is generated by the ADHM matrix

$$\Delta = \frac{\omega}{\sqrt{3}} \begin{bmatrix} \mathbf{1} & i\sigma_3 & -i\sigma_2 \\ 0 & i\sigma_3 & i\sigma_2 \\ i\sigma_3 & 0 & i\sigma_1 \\ i\sigma_2 & i\sigma_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}. \quad (28)$$

A tetrahedrally-symmetric 3-instanton can also be obtained in JNR form, and its holonomy was used to approximate the minimum-energy 3-Skyrmion [5].

For the final example, we consider 4-calorons with cubic symmetry (so G is the 24-element octahedral group). The Nahm data in [8] and [27] do not satisfy (14),

and so we have to change to a basis in which (14) holds. Define

$$\begin{aligned}\Sigma_1 &= - \begin{bmatrix} -\sqrt{3} & 0 & -i & -1 \\ 0 & \sqrt{3} & -1 & i \\ i & -1 & -\sqrt{3} & 0 \\ -1 & -i & 0 & \sqrt{3} \end{bmatrix}, & \Sigma_2 &= - \begin{bmatrix} 0 & \sqrt{3} & 1 & -i \\ \sqrt{3} & 0 & -i & -1 \\ 1 & i & 0 & \sqrt{3} \\ i & -1 & \sqrt{3} & 0 \end{bmatrix}, \\ \Sigma_3 &= - \begin{bmatrix} 2 & -i & 0 & 0 \\ i & 2 & 0 & 0 \\ 0 & 0 & -2 & -i \\ 0 & 0 & i & -2 \end{bmatrix}, & S_1 &= -2 \begin{bmatrix} \sqrt{3} & 0 & -4i & 1 \\ 0 & -\sqrt{3} & 1 & 4i \\ 4i & 1 & \sqrt{3} & 0 \\ 1 & -4i & 0 & -\sqrt{3} \end{bmatrix}, \\ S_2 &= -2 \begin{bmatrix} 0 & -\sqrt{3} & -1 & -4i \\ -\sqrt{3} & 0 & -4i & 1 \\ -1 & 4i & 0 & -\sqrt{3} \\ 4i & 1 & -\sqrt{3} & 0 \end{bmatrix}, & S_3 &= -4 \begin{bmatrix} -1 & -2i & 0 & 0 \\ 2i & -1 & 0 & 0 \\ 0 & 0 & 1 & -2i \\ 0 & 0 & 2i & 1 \end{bmatrix}.\end{aligned}$$

Then $T_j(s) = x(s)\Sigma_j + y(s)S_j$, where

$$y = \frac{\omega_2}{10\wp'(u)}, \quad x = [5\wp(u)^2 - 3]y, \quad (29)$$

with $\omega_2 = (1+i)\Gamma(1/4)^2/(4\sqrt{2\pi})$ and $u = \omega_2(s+1)/2$. Here \wp is the Weierstrass p-function satisfying $\wp'(u)^2 = 4\wp(u)^3 - 4\wp(u)$. The condition (14) follows from the relations

$$\begin{bmatrix} x(-s) \\ y(-s) \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 & -16 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x(s) \\ y(s) \end{bmatrix}. \quad (30)$$

Then, as before, (15) has a unique solution

$$W = \lambda[\mathbf{1} \quad i\sigma_3 \quad i\sigma_1 \quad i\sigma_2], \quad \text{where } \lambda = \sqrt{2x(\pi/\beta) + 16y(\pi/\beta)}; \quad (31)$$

and one may check explicitly that (20) is satisfied for each element of the octahedral group. So here we have a one-parameter family of octahedrally-symmetric 4-calorons, interpolating between the cubic (octahedrally-symmetric) 4-monopole and an octahedrally-symmetric 4-instanton. This instanton is generated by the ADHM matrix

$$\Delta = \frac{|\omega_2|}{\sqrt{2}} \begin{bmatrix} 1 & i\sigma_3 & i\sigma_1 & i\sigma_2 \\ \frac{\sqrt{3}}{2}i\sigma_1 - i\sigma_3 & -\frac{\sqrt{3}}{2}i\sigma_2 & -\frac{1}{2}i\sigma_2 & \frac{1}{2}i\sigma_1 \\ -\frac{\sqrt{3}}{2}i\sigma_2 & -\frac{\sqrt{3}}{2}i\sigma_1 - i\sigma_3 & \frac{1}{2}i\sigma_1 & \frac{1}{2}i\sigma_2 \\ -\frac{1}{2}i\sigma_2 & \frac{1}{2}i\sigma_1 & \frac{\sqrt{3}}{2}i\sigma_1 + i\sigma_3 & -\frac{\sqrt{3}}{2}i\sigma_2 \\ \frac{1}{2}i\sigma_1 & \frac{1}{2}i\sigma_2 & -\frac{\sqrt{3}}{2}i\sigma_2 & -\frac{\sqrt{3}}{2}i\sigma_1 + i\sigma_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{bmatrix}, \quad (32)$$

which may be compared with the symmetric 4-instanton example described in [5].

In conclusion, we have seen that, at least for charge $N \leq 4$, there is an intimate connection between symmetric monopoles, symmetric calorons, symmetric instantons, and (via holonomy) Skyrmons. Many open questions remain, of which the following are a few.

- Several more symmetric monopoles (of higher charge) are known — do all of these arise as limiting cases of calorons with the same symmetry? More generally, is it true that any symmetric monopole has to be a special case of a symmetric caloron?
- Similarly, does every symmetric instanton [6] extend to a family of symmetric calorons? Note that such families are much more general, in that there may not be a symmetric monopole at the ‘other end’.
- What is the role of harmonic maps, which are known to be related to symmetric monopoles and Skyrmons [9]? Does this involve the interpretation of calorons as monopoles with a loop group as their gauge group [28, 29]?

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